Bianchi Type-V model in R^n-gravity: A dynamical systems approach

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Abstract. The accelerated expansion of the universe and the rotational dynamics of galaxies have become part of the mysteries of the physical world and have had theorists working tirelessly in the past years. There is no consensus on what is causing these observable effects: whether it is the yet-to-be-discovered dark energy and dark matter or it is the breaking down of our currently accepted theory of gravity, General Relativity, on larger scales. In this paper, we assume it is the latter and analyze R^n-gravity - a type of modified theory of gravity - in the Bianchi Type-V spacetime. Numerous accelerating solutions are found and their stability is analyzed. There is one particular solution that was found to be stable for a wide range of values of n and it makes for a possible solution for the accelerated expansion anomaly.

1. Introduction

The dynamics of the cosmos at large are nothing like any system we have encountered in the solar system. Experimental evidence indicate that galaxies are moving apart instead of pulling together as gravity would have them [1, 2]. To account for this, an energy density with a negative pressure is included into the energy budget of the universe when calculations are being carried out in the standard model of cosmology [3]. This energy cannot be accounted for using any of our most successful theoretical tools, in fact, the results are disastrous. In the frame work of quantum field theories, the expansion was attributed to the vacuum energy which observations have shown to have a density of not more than $10^{-29}\mathrm{g/cm}^3$ [4]. The theoretical calculations, however, exceeds this bound by 55 orders of magnitude [4].

There is a search for a theory that contains the undisputed General Relativity as a subset and, at the same time, explain why is spacetime expanding. Various candidates have been put forward, ranging from quantum field theories to the so-called modified theories of gravity [5]. One of the promising of the modified theories is R^n-gravity, obtained from replacing the Ricci scalar R in the Einstein-Hilbert action (in normalized units) [6, 7, 8, 9]

$$S_{EH} = \frac{1}{2}\int dx^4\sqrt{-g}(R + 2\mathcal{L}_M), \tag{1}$$

with R^n such that the generalized action is

$$S = \frac{1}{2}\int dx^4\sqrt{-g}(R^n + 2\mathcal{L}_M). \tag{2}$$

Here g is the determinant of the metric tensor and $\mathcal{L}_M$ is the matter Lagrangian. Varying the action (2) with respect to the metric, performing a 1+3 covariant decomposition to the resulting field equations, and imposing the Bianchi V group of isometries onto the underlying spacetime, we obtain the field equation for R^n-gravity in Bianchi Type V spacetime for a non-tilted fluid:

$$
\begin{aligned}
\dot{a} &= -\frac{1}{3}a\Theta, \\
\dot{\sigma} &= -\Theta\sigma - (n-1)\frac{\dot{R}}{R}\sigma, \\
\dot{\Theta} &= \frac{R}{2n} + (n-1)\frac{\dot{R}}{R}\Theta - \frac{\rho}{nR^{n-1}} - \frac{1}{3}\Theta^2 - 2\sigma^2, \\
\dot{\rho} &= -\Theta(1+\omega)\rho, \\
0 &= \frac{1}{3}\Theta^2 - 3a^2 - \sigma^2 - \frac{(n-1)}{2n}R - \frac{\rho}{nR^{n-1}} + (n-1)\frac{\dot{R}}{R}\Theta.
\end{aligned}
\tag{3}
$$

Here σ is the shear scalar, ρ is the matter density, Θ is the rate of expansion scalar, ω is the equation of state parameter, and a is a parameter related to the spatial curvature scalar, $\tilde{R}$, by the formula $\tilde{R} = -6a^2$ [3, 10, 11].

2. Analysis

Defining the expansion-normalized variables [7, 9]:

$$
\begin{aligned}
d\tau &:= \Theta dt, & \Sigma &:= \frac{3\sigma^2}{\Theta^2}, & W &:= \frac{9a^2}{\Theta^2}, \\
x &:= \frac{3\dot{R}}{R\Theta}(n-1), & y &:= \frac{3R}{2n\Theta^2}(n-1), & z &:= \frac{3\rho}{nR^{n-1}\Theta^2},
\end{aligned}
\tag{4}
$$

and substituting into (3), we obtain the dimensionless equations:

$$
\begin{aligned}
W' &= \frac{2W}{3}\left(1 - \frac{n}{n-1}y - W + \Sigma\right), \\
\Sigma' &= -\frac{2\Sigma}{3}\left[\left(\frac{2n-1}{n-1}\right)y + z - 2W\right], \\
y' &= \frac{y}{3(n-1)}\left[(3-2n)W + (2n-1)\Sigma - (2n-1)y + z + (4n-5)\right], \\
z' &= \frac{z}{3}\left[-z + (2-3\omega) - 3W + \Sigma - \left(\frac{3n-1}{n-1}\right)y\right].
\end{aligned}
\tag{5}
$$

For brevity we focus our attention on the vacuum solutions, $z = 0$. The fixed points and the corresponding solutions are summarized in tables 1 and 2, respectively. Of all the obtained solutions, only the solution at the fixed point $\mathcal{B}$ could produce an accelerating spacetime depending on the value of n. Mathematically, the slope, $\dot{l}(t)$, and the concavity, $\ddot{l}(t)$, of the scale factor, $l(t)$, are always positive for accelerating solutions. For the scale factor in $\mathcal{B}$, this is true for $n \in \left(-\infty, \frac{1-\sqrt{3}}{2}\right) \cup \left(\frac{1+\sqrt{3}}{2}, 2\right)$.

Stability analysis [12, 13] show that this fixed point is a stable node for $n \in \left(-\infty, \frac{1-\sqrt{3}}{2}\right) \cup \left(\frac{1}{2}, 1\right) \cup \left(\frac{1+\sqrt{3}}{2}, \infty\right)$, a saddle for $n \in \left(\frac{1-\sqrt{3}}{2}, \frac{1}{2}\right) \cup \left(\frac{5}{4}, \frac{1+\sqrt{3}}{2}\right)$ and an unstable node for $1 < n < \frac{5}{4}$. Comparing the values of n such that $\mathcal{B}$ is stable and those which result in accelerating solutions, it is

Table 1. Local fixed points for the vacuum solutions.

	(Σ_i, W_i, y_i)
Point $\mathcal{A}$	$(0,1,0)$
Point $\mathcal{B}$	$\left(0,0,\frac{4n-5}{2n-1}\right)$
Point $\mathcal{C}$	$[0,-2n^2+2n+1,2(n-1)^2]$
Point $\mathcal{D}$	$\left[-\frac{n(4n-5)}{2(2n-1)},-\frac{n-2}{2},-\frac{(n-1)(n-2)}{2n-1}\right]$
Line $\mathscr{L}_1$	$(\Sigma_*,0,0), \quad \Sigma_* \geq 0$

Table 2. Vacuum solutions at each fixed point.

	$a(t)$	$\sigma(t)$	$R(t)$	$l(t)$
Point $\mathcal{A}$	$a=a_0(t-t_0)^{-1}$	0	0	$l(t)=l_0(t-t_0)$
Point $\mathcal{B}$	0	0	$R=\frac{6n(4n-5)(n-1)(2n-1)}{(n-2)^2t^2}$	$l(t)=l_0(t-t_0)^{\frac{(2n-1)(1-n)}{n-2}}$
Point $\mathcal{C}$	$a=a_0(t-t_0)^{-1}$	0	$R=\frac{12n(n-1)}{t^2}$	$l(t)=l_0(t-t_0)$
Point $\mathcal{D}$	$a=a_0(t-t_0)^{-1}$	$\sigma=\sigma_0(t-t_0)^{2n-5}$	$R=\frac{6n(2-n)}{(2n-1)t^2}$	$l(t)=l_0(t-t_0)$
Line $\mathscr{L}_1$	0	$\sigma=\sigma_0(t-t_0)^{-\frac{2+\Sigma_*}{1+2\Sigma_*}}$	0	$l(t)=l_0(t-t_0)^{\frac{1}{1+2\Sigma_*}}$

seen that $\mathcal{B}$ is stable for all values of n that result in accelerating solutions. This implies that any initial conditions in the vicinity of $\mathcal{B}$ will have solutions which qualitatively behave like those of $\mathcal{B}$ [12]. Roughly speaking, this means that a universe which has an initial state similar to that described by the solutions of $\mathcal{B}$, will evolve to have characteristics which are similar to those of $\mathcal{B}$.

The phase space is unbounded and asymptotic fixed points must be considered to complete the analysis. Performing the following change of variables:

$$\begin{aligned}\Sigma &= \bar{r}\cos\phi\sin\theta,\\ W &= \bar{r}\sin\phi\sin\theta,\\ y &= \bar{r}\cos\theta,\end{aligned} \tag{6}$$

where $\bar{r}=\frac{r}{1-r}$ [7, 14], $0<\phi\leq\frac{\pi}{2}$ and $0<\theta<\pi$, and taking the limit $\bar{r}\to\infty$ $(r\to 1)$ the equations in (5) reduce to:

Table 3. Asymptotic fixed points and solutions for the scale factor.

Point	(ϕ_i, θ_i)	Scale factor
$\mathcal{A}_\infty$	$(0,0)$	$\lvert\tau-\tau_\infty\rvert=\left[C_1 \pm C_0\left\lvert\frac{n-1}{2n-1}\right\rvert(t-t_0)\right]^{\frac{2n-1}{n-1}}$
$\mathcal{B}_\infty$	$(0,\frac{\pi}{2})$	$\tau-\tau_\infty=C_1\ln\lvert t-t_0\rvert+C_2$
$\mathcal{C}_\infty$	$(\frac{\pi}{2},0)$	$\lvert\tau-\tau_\infty\rvert=\left[C_1 \pm C_0\left\lvert\frac{n-1}{2n-1}\right\rvert(t-t_0)\right]^{\frac{2n-1}{n-1}}$
$\mathcal{D}_\infty$	$(\frac{\pi}{2},\frac{\pi}{2})$	$\lvert\tau-\tau_\infty\rvert=\left[C_1\pm\frac{1}{2}C_0(t-t_0)\right]^2$
$\mathcal{E}_\infty$	$\left[\arctan\left(\frac{1}{3}\right),\frac{\pi}{2}\right]$	$\lvert\tau-\tau_\infty\rvert=\left[C_1\pm\frac{1}{2}C_0(t-t_0)\right]^2$

Table 4. Asymptotic fixed points and solutions for the scale factor.

Point	$a(\tau)$	$\sigma(\tau)$	$R(\tau)$
$\mathcal{A}_\infty$	0	0	$R=R_0\frac{n}{2n-1}\lvert\tau-\tau_\infty\rvert^{\frac{1}{2n-1}}$
$\mathcal{B}_\infty$	0	$\sigma=\sigma_0 e^{\tau-\tau_\infty}$	0
$\mathcal{C}_\infty$	0	$\sigma=\sigma_0\sqrt{\frac{n-1}{2n-1}}\lvert\tau-\tau_\infty\rvert^{\frac{1}{2(2n-1)}}$	$R=R_0\frac{n}{2n-1}\lvert\tau-\tau_\infty\rvert^{\frac{2n}{2n-1}}$
$\mathcal{D}_\infty$	a_0	0	0
$\mathcal{E}_\infty$	a_0	σ_0	0

$$r' = -\frac{\sin^2\theta}{3}\left\{2\cos^2\phi[\cos\theta+(\cos\phi-3\sin\phi)\sin\theta]+\frac{1}{n-1}[\cos\theta+(\sin\phi+\cos\phi)\sin\theta]\right\}$$
$$+\frac{1}{3}\left[\frac{3-2n}{n-1}\sin\phi\sin\theta+\frac{2n-1}{n-1}\cos\phi\sin\theta-\frac{2n-1}{n-1}\cos\theta\right], \tag{7}$$

$$\theta' = -\frac{\sin 2\theta}{6(1-r)}\left\{2\cos^2\phi[\cos\theta+(\cos\phi-3\sin\phi)\sin\theta]+\frac{1}{n-1}[\cos\theta+(\sin\phi+\cos\phi)\sin\theta]\right\}, \tag{8}$$

$$\phi' = \frac{\sin 2\phi}{3(1-r)}\left[\cos\theta+(\cos\phi-3\sin\phi)\sin\theta\right]. \tag{9}$$

The fixed points and the solutions are given in tables 3 and 4. The stability of the fixed points is summarized in table 5 and none of them are stable as can be seen therein.

Table 5. Stability of the asymptotic fixed points in vacuum

Point	Eigenvalues	r'	$n < 1/2$	$1/2 < n < 1$	$n > 1$
$\mathcal{A}_\infty$	$\left[\frac{2}{3}, -\frac{2n-1}{3(n-1)}\right]$	$-\frac{2n-1}{3(n-1)}$	Saddle	Saddle	Saddle
$\mathcal{B}_\infty$	$\left[\frac{2}{3}, \frac{2n-1}{3(n-1)}\right]$	0	Unstable	Saddle	Unstable
$\mathcal{C}_\infty$	$\left[-\frac{2}{3}, -\frac{1}{3(n-1)}\right]$	$-\frac{2n-1}{3(n-1)}$	Saddle	Saddle	Saddle
$\mathcal{D}_\infty$	$\left[2, \frac{1}{3(n-1)}\right]$	$-\frac{2}{3}$	Saddle	Saddle	Unstable
$\mathcal{E}_\infty$	$\left[-\frac{\sqrt{10}}{5}, \frac{2\sqrt{10}}{15(n-1)}\right]$	$\frac{2\sqrt{10}}{15}$	Stable	Stable	Saddle

3. Conclusion

Although Bianchi models are generally anisotropic, the accelerating solution discussed describe a spatially flat and anisotropic universe . And since this is a vacuum solution, the acceleration is not influenced by any energy but is an intrinsic property of spacetime. Furthermore, due to the stability of the solution, the universe whose gravitational interaction is governed by R^n-gravity is more likely to asymptotically approach a flat, isotropic and expanding state. This is indeed our own universe. It was shown in [8] that for $n = 1.4$ and $n = 0.6$ the R^n model does mimic the real universe in the context of the FLRW metric, however. The model discussed in this paper allows for anisotropies in the early universe and thus is more general than the model discussed in [8].

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